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On the fragmentary complexity of symbolic sequences

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Abstract

A measure of the ability of a symbolic sequence to be covered by initial fragments of another symbolic sequence is introduced and its basic properties are investigated. Applications to the characterization of symbolic sequences associated with shift mappings on a torus corresponding to a special partitioning of the torus and to multirate systems of coprocessors are considered.

0. Introduction

The classical methods of symbolic dynamics involve a shift operator on a space of infinite symbolic sequences with elements from a finite alphabet. Since dynamical systems can often be reduced to such shift operators, the complexity of a dynamical system can then be characterized by that of a shift action on symbolic sequences. A commonly used measure of such complexity is the ability of a sequence to be covered by finite words from a universal totality that does not depend on the system being investigated. The classical notions of entropy of a dynamical system [8] and linear complexity of sequences [7] find a natural description within this framework.

In this paper a different, but related concept of complexity of infinite sequences based on [4, 5] will be studied. In particular, symbolic sequences $\mathbf{T}$ which can be partitioned (perhaps, excluding an initial fragment) as the adjoint union of initial fragments of another sequence $\mathbf{U}$ will be considered and called $\mathbf{U}$-generated or self-similar. Theorems 1 and 2 show that they do have a self-similar, fragmentary

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structure as commonly understood. The complexity of such a sequence $\mathbf{T}$ with respect to another sequence $\mathbf{U}$ will be estimated by the minimal number $C_f(\mathbf{T},\mathbf{U})$ of samples of arbitrary long initial fragments of the sequence $\mathbf{U}$ that can cover the sequence $\mathbf{T}$ disjointly, and its minimum $C_f(\mathbf{T})$ over all $\mathbf{U}$ will be called the *fragmentary complexity* of $\mathbf{T}$ here. A precise definition and its main properties will be given in Section 1.

It has been shown that the fragmentary complexity measure of sturmian sequences [9, 10] with irrational frequencies, such as the symbolic sequences representing shifts of the unit circle, is equal to 2. In Secton 3 it will be shown that the properties of individual trajectories of multidimensional tori shifts of dimension higher than 2 changes drastically, with the fragmentary complexity measure generally being infinite in such cases (Theorem 4). The situation for 2-dimensional tori shifts is still unclear.

Self-similar fragmentary sequences arise naturally in applications such as the stability analysis of asynchronous systems [5, 10] (see also Theorem 3) or, for example, multirate coprocessors, as well as in the description of self-similar and chaotic phenomena. As seen from [1, 3, 4] this stability problem reduces to that of the nonautonomous difference equation

$$x(n+1) = f[\lambda; n, x(n)], \quad n = 0, 1, 2, \ldots, \tag{1}$$

with the right-hand side $f(\lambda; n, x)$ nonperiodic in n and depending on a parameter λ such that the number of different mappings in $\{f[\lambda; n, \cdot]\}$ is finite. Here the order of different mappings in the sequence $\{f[\lambda; n, \cdot]\}$ corresponds to the order of symbols in a symbolic sequence generated by a certain shift mapping of a torus with a special partitioning. Using a concept similar to the fragmentary complexity of these symbolic sequences it was proved in [4, 5] that the asymptotic stability of Eq. (1) for one particular value of the parameter λ implies its stability for the other values of λ.

1. A measure of fragmentary complexity

1.1. Weakly decomposable texts

Following [7] we shall use linguistic terminology and notation. In particular, elements in symbolic sequences will not be separated by commas. Let $\mathscr{A}$ be a fixed alphabet, that is a set of elements called *letters* or *symbols*. A finite cortege $\mathbf{w} = a_1 \ldots a_n$ of letters from $\mathscr{A}$ is called *a word*, for any words $\mathbf{w}^1 = a_1^1 \ldots a_{n_1}^1$ and $\mathbf{w}^2 = a_1^2 \ldots a_{n_2}^2$ their *product* is the word $\mathbf{w}^1\mathbf{w}^2 = a_1^1 \ldots a_{n_1}^1 a_1^2 \ldots a_{n_2}^2$, and the *left factor* (*of the length* $j \leqslant n$) of the word $\mathbf{w} = a_1 \ldots a_n$ is the initial fragment $\mathbf{w}(j) = a_1 \ldots a_j$ of $\mathbf{w}$. An infinite sequence $\mathbf{T} = a_1 a_2 \ldots$ from the alphabet $\mathscr{A}$ is called *an infinite word* or *text*, the word $\mathbf{T}(n) = a_1 a_2 \ldots a_n$ its *left factor* (*of the length* n) and the text $a_{n+1} a_{n+2} \ldots$ its *right factor* (*of the colength* n), while any word $a_i \ldots a_j$ with $i \leqslant j$ is called a *factor* of $\mathbf{T}$.

An ordered finite set

$$\mathbf{S} = \{\mathbf{w}_1, \ldots, \mathbf{w}_\nu\} \tag{2}$$

of words of length $l_1, \ldots, l_\nu$ is said to be *generating* if it satisfies the following properties:

(P1) $0 < l_1 < \cdots < l_\nu$.

(P2) The word $\mathbf{w}_\iota$ is a left factor of $\mathbf{w}_\nu$ for each $\iota = 1, \ldots, \nu - 1$, that is, $\mathbf{w}_\iota$ coincides with the initial segment of $\mathbf{w}_\nu$ of length l_ι.

A finite or infinite word $\mathbf{w}$ is **S**-*decomposable* if it can be represented as a product of words belonging to a set of words (2), while a text **T** is *weakly* **S**-*decomposable* if it has an **S**-decomposable right factor. In other words, **T** is weakly **S**-decomposable if there exists an increasing sequence $\boldsymbol{d} = \{d_0, d_1, d_2, \ldots\}$ of natural numbers such that $r_i = d_i - d_{i-1}$ is equal to one of the numbers l_ι, for $\iota = 1, \ldots, \nu$ and $\mathbf{w}_i = a_{d_{i-1}} \ldots a_{d_{i-1}}$; such a sequence $\mathbf{d}$ is *a weak* **S**-*decomposition* of **T**.

Now consider two texts **T** and **U**. The text **T** will be called a **U**-*generated* if for any N there exists a finite generating set **S** of left factors of the text **U** such that all words $\mathbf{w} \in \mathbf{S}$ are of length greater than N and the text **T** is weakly **S**-decomposable. A periodic text **T** is clearly **T**-generated, or *self-generative* where **U** is a periodic part of **T**, but as will be seen in Section 1.3 that there also exist self-generative texts with much more complicated structure. The fact that a text **T** is **U**-generated for a certain text **U** can be useful. For example, if a text **U** is ergodic in the sense that for all $a \in \mathscr{A}$ the frequencies $\lim_{n \to \infty} q_n(a)$ exist, where $q_n(a)$ is the number of times the letter a occurs in $\mathbf{U}(n)$, then any **U**-generated text **T** is also ergodic with the same limiting frequencies. This idea was used in the stability analysis of asynchronous systems including multirate and frequency desynchronized systems [4, 5].

Denote by $\mathscr{S}(\mathbf{T}, \mathbf{U})$ the family of all finite generating sets **S** of left factors of the text **U** for which the text **T** is weakly **S**-decomposable and by $\mathscr{S}_*(\mathbf{T}, \mathbf{U})$ the totality of elements of $\mathscr{S}(\mathbf{T}, \mathbf{U})$ of the form (2) which satisfy the additional property:

(P3) For each $\iota = 1, \ldots, \nu - 1$ the word $\mathbf{w}_\iota$ is not a power, that is, cannot be partitioned into repeating fragments.

Theorem 1. *Let a text* **T** *be* **U**-*generated. Let* $\mathbf{S}^{\text{short}} \in \mathscr{S}_*(\mathbf{T}, \mathbf{U})$, $\mathbf{S}^{\text{long}} \in \mathscr{S}(\mathbf{T}, \mathbf{U})$ *and suppose that the shortest word from* $\mathbf{S}^{\text{long}}$ *is longer than the longest word from* $\mathbf{S}^{\text{short}}$. *Then every word from* $\mathbf{S}^{\text{long}}$ *is* $\mathbf{S}^{\text{short}}$-*decomposable and each weak* $\mathbf{S}^{\text{long}}$-*decomposition* $\mathbf{d}^{\text{long}}$ *is a subset of any weak* $\mathbf{S}^{\text{short}}$-*decomposition* $\mathbf{d}^{\text{short}}$ *satisfying* $d_0^{\text{short}} \leqslant d_0^{\text{long}}$.

Proof. Suppose the opposite. Then there exists a number $d \in \mathbf{d}^{\text{long}}$ and an index I such that $d_I^{\text{short}} < d < d_{I+1}^{\text{short}}$. Write $\mathbf{w}^1 = a_{d_I^{\text{short}}} \ldots a_{d-1}$ and $\mathbf{w}^2 = a_d \ldots a_{d_{I+1}^{\text{short}} - 1}$. By property (P1) and the assumtions of the theorem we have $\mathbf{w}^1\mathbf{w}^2 = \mathbf{w}^2\mathbf{w}^1$. Hence, by [7, Proposition 1.3.2] the word $\mathbf{w} = a_{d_I^{\text{short}}} \ldots a_{d_{I+1}^{\text{short}} - 1}$ is a power. By the construction, this word belongs to a generating set, but this contradicts property (P3) of $\mathscr{S}_*(\mathbf{T}, \mathbf{U})$. $\square$

Informally speaking, Theorem 1 says that every **U**-decomposition can be considered as the result of a partitioning of some "bigger" **U**-decomposition.

Example 1. Let $\mathscr{A} = \{a, b\}$,

$$\mathbf{U} = abbababb\ldots, \qquad \mathbf{T} = babbabbababbababbabbab\ldots$$

and let $\mathbf{S}^{\text{short}} = \{\mathbf{w}_1^{\text{short}}, \mathbf{w}_2^{\text{short}}\}$, $\mathbf{S}^{\text{long}} = \{\mathbf{w}_1^{\text{long}}, \mathbf{w}_2^{\text{long}}\}$, where

$$\mathbf{w}_1^{\text{short}} = ab, \quad \mathbf{w}_2^{\text{short}} = abb, \quad \mathbf{w}_1^{\text{long}} = abbab, \quad \mathbf{w}_2^{\text{long}} = abbababb.$$

Then the following decomposition of **T** is valid:

$$\mathbf{T} = b\ \underbrace{abb}_{\mathbf{w}_2^{\text{short}}}\ \overbrace{\underbrace{abb}_{\mathbf{w}_2^{\text{short}}}\ \underbrace{ab}_{\mathbf{w}_1^{\text{short}}}}^{\mathbf{w}_1^{\text{long}}}\ \overbrace{\underbrace{abb}_{\mathbf{w}_2^{\text{short}}}\ \underbrace{ab}_{\mathbf{w}_1^{\text{short}}}\ \underbrace{abb}_{\mathbf{w}_2^{\text{short}}}}^{\mathbf{w}_2^{\text{long}}}\ \overbrace{\underbrace{abb}_{\mathbf{w}_2^{\text{short}}}\ \underbrace{ab}_{\mathbf{w}_1^{\text{short}}}}^{\mathbf{w}_1^{\text{long}}}\ \ldots$$

The text **U** will be called *self-generative* if for any N there exists a finite generating set **S** of left factors of **U** itself such that all words $\mathbf{w} \in \mathbf{S}$ are of length greater than N and the text **U** is **S**-decomposable. As a corollary to Theorem 1 we have:

Corollary 1. *A text* **U** *is self-generative if and only if there exists a* **U***-generated text* **T**.

Proof. If the text **T** is periodic after a certain index N then text **U** must be also periodic and there is nothing to prove. Consider the case when the text **T** is not eventually periodic. Let $\mathbf{S} \in \mathscr{S}_*(\mathbf{T}, \mathbf{U})$ be a generating set for the text **T** consisting of v left factors $\mathbf{U}(l_1), \ldots, \mathbf{U}(l_v)$. The corollary will be proven if we establish that the text **U** is **S**-decomposable. Consider the sequence of originating for **T** sets

$$\mathbf{S}_n = (\mathbf{w}_1, \ldots, \mathbf{w}_{v(n)}), \tag{3}$$

which satisfy the following conditions:

(Q1) Each element of any set $\mathbf{S}_n$ is a left factor of **U**.

(Q2) The length of the shortest word in $\mathbf{S}_n$ is greater than n.

By Theorem 1 for $n \geqslant l_v$ all words from the set (3) are **S**-decomposable. Denote the corresponding decomposition by

$$\mathbf{d}^{n,\imath} = d_1^{n,\imath}, \ldots, \delta_{m(n,\imath)}^{n,\imath}), \quad \imath = 1, \ldots, v(n) \tag{4}$$

and denote by $\mathbf{d}^*$ the sequence that is a limit point of sequence (4) in the topology of point-wise convergence. By the construction, the sequence $\mathbf{d}^*$ is a **S**-decomposition of U, and the corollary is proven. □

Self-generative texts have an important property of being recurrent. For each text **T** denote by $\mathscr{W}(\mathbf{T}, n)$ the totality of words $a_i a_{i+1} \ldots a_{i+n-1}$, $i = 1, 2, \ldots$ A text **T** is said to be *recurrent* [10] if for each natural number m there exists a natural number n such that any word from $\mathscr{W}(\mathbf{T}, m)$ is a factor of other words from $\mathscr{W}(\mathbf{T}, n)$.

Lemma 1. *Each self-generative text* $\mathbf{U}$ *is recurrent.*

Proof. Choose a natural number N such that all words from $\mathscr{W}(\mathbf{U}, m)$ are factors of $\mathbf{U}(N)$. Consider a generating set $\mathbf{S}$ of left factors of $\mathbf{U}$ such that $\mathbf{U}$ is $\mathbf{S}$-decomposable and all words from $\mathbf{S}$ are longer than N. Let L denote the length of longer word in $\mathbf{S}$. By construction every word from $\mathscr{W}(\mathbf{U}, m)$ is a factor of each word from $\mathscr{W}(\mathbf{U}, L)$, and so the lemma is proven. □

The general construction of self-generative texts to be presented in Section 1.3 thus provides a means of constructing recurrent texts.

1.2. Fragmentary complexity of texts

Let a text $\mathbf{T}$ be $\mathbf{U}$-generated. Denote by $\mathscr{S}(\mathbf{T}, \mathbf{U}; N)$ the subset of $\mathscr{S}(\mathbf{T}, \mathbf{U})$ containing those generating sets $\mathbf{S}$ all words from which are longer than N. For any natural number N define by $C_f(\mathbf{T}, \mathbf{U}; N)$ the minimal quantity $C_f(\mathbf{T}, \mathbf{U}; N)$ of elements in sets from $\mathscr{S}(\mathbf{T}, \mathbf{U}; N)$. Clearly, $C_f(\mathbf{T}, \mathbf{U}; N)$ is increasing as a function N. It is naturally to characterize the complexity of the text $\mathbf{T}$ with respect to the text $\mathbf{U}$ by the rate of increase of this function.

In particular, of a special interest is the situation when this function is bounded, in which case we will call the number

$$C_f(\mathbf{T}, \mathbf{U}) = \max_N C_f(\mathbf{T}, \mathbf{U}; N) \tag{5}$$

the $\mathbf{U}$-*complexity* of the text $\mathbf{T}$. It is convenient to set $C_f(\mathbf{T}, \mathbf{U}) = \infty$ if the function $C_f(\mathbf{T}, \mathbf{U}; N)$ is unbounded or if $\mathbf{T}$ is not $\mathbf{U}$-generated. If the text $\mathbf{T}$ has a finite $\mathbf{U}$-complexity with respect to at least one text $\mathbf{U}$ then define $C_f(\mathbf{T}) = \min_{\mathbf{U}} C_f(\mathbf{T}, \mathbf{U})$. This quantity $C_f(\mathbf{T})$ will be called the *fragmentary complexity of* $\mathbf{T}$.

1.3. General construction of self-generative texts

We now describe a general construction of self-generative texts with fragmentary complexity not exceeding C. Let $\mathscr{A}_k$ be an alphabet with $k > 1$ letters, say $1, \ldots, k$. If to every letter $\lambda \in \mathscr{A}_k$ there corresponds a word $\mathbf{w} = F(\lambda) \in \mathscr{W}(\mathscr{A})$, then to each word $\mathbf{v}$ of the alphabet $\mathscr{A}_k$ we associate a word $F(\mathbf{v})$ obtained by substituting the word $F(\lambda)$ for each letter λ in the word $\mathbf{v}$.

Let us now choose

- a natural number $v \leqslant C$,
- a sequence $\mathscr{S}$ of generating sets $\mathbf{S}_n$, $n = 1, 2, \ldots$ in the alphabet $\mathscr{A}_{v(n-1)}$ containing words $\mathbf{v}^{n,\iota}$, $\iota = 1, \ldots, v(n)$, with lengths $l(\mathbf{v}^{n,\iota}) > n$;

- a particular generating set $\mathbf{S}_0^*$ of words of the alphabet $\mathscr{A}$ which contains ν elements.

Then we construct recursively the generating subsets

$$\mathbf{S}_n^* = \{\mathbf{w}^{n,1}, \ldots, \mathbf{w}^{n,\nu(n)}\}, \quad n = 1, 2, \ldots,$$

in the alphabet $\mathscr{A}$. Suppose that $\mathbf{S}_{n-1}^*$ is already defined. Then define $F_n(\lambda) = \mathbf{w}^{n-1,\lambda}$ for $\lambda = 1, \ldots, \nu(n-1)$ and set $\mathbf{w}^{n,\iota} = F_n(\mathbf{v}^{n,\iota})$, $\iota = 1, \ldots, \nu(n)$.

Example 2. Let $\mathscr{A} = \{a, b\}$, $S_0^* = (a, ab)$ and

$$\mathbf{S}_1 = \{1, 12\}, \quad \mathbf{S}_2 = \{21, 211\}, \quad \mathbf{S}_3 = \{121, 1211\}.$$

Then

$$F_1(1) = a, \; F_1(2) = ab \quad \text{and} \quad \mathbf{S}_1^* = \{\underbrace{a}_{1}, \underbrace{a}_{1}\underbrace{ab}_{2}\}.$$

Analogously,

$$F_2(1) = a, \; F_2(2) = aab \quad \text{and} \quad \mathbf{S}_2^* = \{\underbrace{\underbrace{a}_{1}\underbrace{ab}_{2}}_{2}\underbrace{\underbrace{a}_{1}}_{1}, \underbrace{\underbrace{a}_{1}\underbrace{ab}_{2}}_{2}\underbrace{\underbrace{a}_{1}}_{1}\underbrace{\underbrace{a}_{1}}_{1}\}.$$

Further,

$$F_3(1) = aaba, \quad F_3(2) = aabaa$$

and

$$\mathbf{S}_3^* = \{\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}}_{1}\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}\underbrace{a}_{1}}_{2}\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}}_{1}, \underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}}_{1}\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}\underbrace{a}_{1}}_{2}\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}}_{1}\underbrace{\underbrace{aab}_{2}\underbrace{a}_{1}}_{1}\}.$$

Clearly, $\mathbf{w}^{n,1}$ is a left factor of $\mathbf{w}^{n+1,1}$ and $\lim_{n\to\infty} l(\mathbf{w}^{n,1}) = \infty$. Therefore there exists a pointwise limit $\mathbf{U} = \mathbf{U}(\nu, \mathscr{S}, \mathbf{S}_0^*)$ of the sequence of words $\mathbf{w}^{n,1}$ when $n \to \infty$.

Lemma 2. *Each text* $\mathbf{U}(\nu, \mathscr{S}, \mathbf{S}_0^*)$ *is self-generative of* U-complexity *not exceeding* ν. *Moreover, each self-generative text* $\mathbf{U}$ *of* U-complexity C *can be regarded as* $\mathbf{U}(C, \mathscr{S}, \mathbf{S}_0^*)$ *for appropriate* $\mathscr{S}$ *and* $\mathbf{S}_0^*$.

Proof. By construction each text $\mathbf{U}(\nu, \mathscr{S}, \mathbf{S}_0^*)$ is self-generative of U-complexity no more than ν. Therefore, we need only prove that each self-generative text $\mathbf{U}$ of U-complexity C coincides with a text $\mathbf{U}(C, \mathscr{S}, \mathbf{S}_0^*)$ for appropriate $\mathscr{S}$ and $\mathbf{S}_0^*$.

Consider the case where the text $\mathbf{U}$ is not periodic. Choose a certain set $\mathbf{S}_0^* = (\mathbf{U}(l_1^0), \ldots, \mathbf{U}(l_C^0)) \in \mathscr{S}_*(\mathbf{U}, \mathbf{U})$ based on $\mathbf{U}$. By definition there exists a sequence of such sets

$$\mathbf{S}_n^* = \{\mathbf{U}(l_1^n), \ldots, \mathbf{U}(l_C^n)\} \in \mathscr{S}_*(\mathbf{U}, \mathbf{U}) \tag{6}$$

for which $l_C^{n-1} \leqslant l_1^n$, $n = 1, 2, \ldots$

By Theorem 1 each word from $\mathbf{S}_n^*$ is $\mathbf{S}_{n-1}^*$-decomposable. Denote the respective decomposition by

$$\mathbf{d}^{n,\iota} = \{d_0^{n,\iota}, d_1^{n,\iota}, \ldots, d_{m,(n,\iota)}^{n,\iota}\}, \quad \iota = 1, \ldots, C,$$

and introduce words $\mathbf{v}^{n,\iota} = v_1^{n,\iota} \ldots v_{m(n,\iota)}^{n,\iota}$, $i = 1, \ldots, C$, in the alphabet $\mathscr{A}_C$ by equalities $v_i^{n,\iota} = \lambda$ if and only if $l_i^{n,\iota} - l_{i-1}^{n,\iota} = l_\lambda^{n-1}$. Define $\mathbf{S}_n = (\mathbf{v}^{n,1}, \ldots, \mathbf{v}^{n,C})$ and $\mathscr{S} = \{\mathbf{S}_1, \mathbf{S}_2, \ldots\}$. Then, by construction, $\mathbf{U} = \mathbf{U}(C, \mathscr{S}, \mathbf{S}_0^*)$, which is the assertion of lemma. $\square$

1.4. Texts with finite fragmentary complexity

For any alphabet $\mathscr{A}_*$ denote by $\mathscr{W}(\mathscr{A}_*)$ the totality of finite words in this alphabet. If there is a word $\mathbf{w} = F(a) \in \mathscr{W}(\mathscr{A}_*)$ for any letter $a \in \mathscr{A}$ then corresponding to the text $\mathbf{T}$ in the alphabet $\mathscr{A}$, denote the text $F(\mathbf{T})$ in the alphabet $\mathscr{A}_*$ be formed by substituting the word $F(a_i)$ for each letter a_i of the text $\mathbf{T}$. The text $\mathbf{T}$ is said to be *eventually periodic* if it has a right factor which is periodic.

Lemma 3. *The following assertions are true*:

(a) *the fragmentary complexity of a text is equal to the fragmentary complexity of any of its right factors*;

(b) *the fragmentary complexity of a text is equal to* 1 *if and only if this text is eventually periodic*;

(c) *for any function* $F: \mathscr{A} \mapsto \mathscr{W}(\mathscr{A}_*)$ *and any text* $\mathbf{T}$ *in the alphabet* $\mathscr{A}$ *the complexity inequality* $\mathscr{C}_f(\mathbf{T}) \geqslant C_f(F(\mathbf{T}))$ *holds.*

Another classical set of "simple" texts is the class of texts with linear complexity for subwords [7]. The text $\mathbf{T}$ is said to be of *linear complexity for subwords* if the number $\#(\mathbf{T}, N)$ of its subwords of length N satisfies the bound

$$\sup_N \frac{\#(\mathbf{T}, N)}{N} < \infty. \tag{7}$$

Generally speaking, the properties of a text "to have finite fragmentary complexity" and "to be of linear complexity for subwords" do not follow one from another. Note that for texts of fragmentary complexity 2 the estimate

$$\lim_{K \to \infty} \inf_{N \geqslant K} \frac{\#(\mathbf{T}, N)}{N} < \infty \tag{8}$$

is always true. This is slightly weaker than (7). Note also that texts with the fragmentary complexity 2 always contain squares, i.e. repeated words one immediately next to other. It is not clear to us if there exist cubic free words of fragmentary complexity 2 (probably, the well known Thue–Morse words [7] are not fragmentary).

Let us describe one more property of texts with fragmentary complexity 2. For any integer $\gamma \geqslant 0$ and any sequence $\mathbf{d}$ denote by $\mathrm{Pr}_\gamma(\mathbf{d})$ the subsequence of $\mathbf{d}$ consisting of elements of d_i with indices $d_i \geqslant \gamma$. Recall also that $\mathrm{U}(i)$ denotes the left factor of the length i of U. Analogously to Theorem 1 it can be shown that:

Theorem 2. *Let a text* $\mathbf{T}$ *have* U*-fragmentary complexity* 2 *and suppose that* $\mathbf{T}$ *is weakly* $(\mathbf{U}(i), \mathbf{U}(j))$*-decomposable where* $(\mathbf{U}(i), \mathbf{U}(j)) \in \mathscr{S}_*(\mathbf{T}, \mathbf{U})$. *Then for any two weak* $(\mathbf{U}(i), \mathbf{U}(j))$*-decompositions* $\mathbf{d}$ *and* $\mathbf{d}^*$ *the identity* $\mathrm{Pr}_L \mathbf{d} = \mathrm{Pr}_L \mathbf{d}^*$ *holds for* $L = \max\{d_0, d_0^*\} + i + j$.

2. Fragmentary complexity of tori shifts

2.1. The one-dimensional case

Consider the mapping f of the interval $[0, 1)$ onto itself defined by

$$f(x) = x + \varphi(x) (\mathrm{mod}\ 1),$$

where φ is a bounded 1-periodic function satisfying

$$|\varphi(x) - \varphi(y)| < |x - y|, \quad x \neq y \text{ (see Fig. 1).}$$

Each point $x \in [0, 1)$ generates a sequence $\{x_n\}$ defined by $x_0 = x$ and the recurrence relation $x_{n+1} = f(x_n)$, $n = 0, 1, \ldots$ The limit

$$\tau(f) = \lim_{n \to \infty} n^{-1} \sum_{k=1}^{k} \psi_k(x),$$

where $\psi_k(x) = \varphi(f^k(0))$, $k = 1, 2, \ldots$, exists and is independent of x. It is called [2] the *rotation number of the mapping f.* If, for instance

$$f(x) = f_\tau(x) = x + \tau (\mathrm{mod}\ 1), \tag{9}$$

where $\tau \in [0, 1)$ is a fixed real number, then $\tau(f) = \tau$.

Suppose that corresponding to each point $x \in [0, 1)$ there is a symbolic sequence (text)

$$\mathbf{T}(x, f) = \sigma_0(x)\sigma_1(x)\ldots\sigma_n(x)\ldots \tag{10}$$

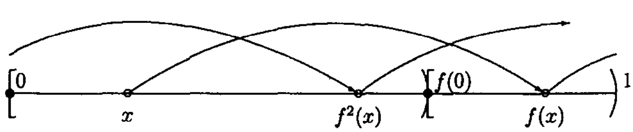

Fig. 1. One-dimensional shift mapping.

consisting of two letters, say a and b, where

$$\sigma_n(x) = \begin{cases} a & \text{if } x_n = f^n(x) \in [0, f(0)), \\ b & \text{if } x_n = f^n(x) \in [f(0), 1). \end{cases} \tag{11}$$

Texts (10) are called as *sturmian beams* with a-frequence $\tau(f)$ in [10]. Note that a different "internal" characterization of sturmian beams is proposed in [10].

If the value τ is rational then all texts (10) are, clearly, eventually periodic and by assertion (b) of Lemma 3 the fragmentary complexity of each text $\mathbf{T}(x,f)$ with $x \in [0,1)$ is equal to 1. The following result regarding the case of irrational τ is a corollary to Theorem 1 of [5] (see also [4, Theorem 5]).

Theorem 3. *Let $\tau(f)$ be irrational and $x \in [0,1)$ with $x \neq f(0)$. Then $\mathbf{T}(x,f)$-fragmentary complexity of each text $\mathbf{T}(y,f)$ with $y \in [0,1)$ is equal to 2.*

2.2. The multidimensional case

The attempts of the authors to formulate an analogue of Theorem 3 for shift mappings of multidimensional tori have not been successful. Nevertheless, some interesting insights into why a direct generalization of this theorem is not possible have been obtained.

Let I^M be the unit multidimensional cube $[0,1) \times [0,1) \times \cdots \times [0,1) = [0,1)^M$ of the space $\mathbb{R}^M$, let $\tau = \{\tau_1, \tau_2, \ldots, \tau_M\}$ be a point in I^M and consider the shift mapping f_τ from the cube I^M onto itself defined by

$$f_\tau(x) = \{x_1 + \tau_1 \,(\mathrm{mod}\, 1), x_2 + \tau_2 \,(\mathrm{mod}\, 1), \ldots, x_M + \tau_M \,(\mathrm{mod}\, 1)\},$$

where $x = \{x_1, x_2, \ldots, x_M\} \in I^M$. In addition, denote by $\mathscr{U}$ the set of all subsets $U_i \subset I^M$, $i = 1, 2, \ldots, 2^M$, of the form $U_i = H_1 \times H_2 \times \cdots \times H_M$ where each H_j coincides either with $[0, \tau_j)$ or with $[\tau_j, 1)$. Finally, let a letter a_i correspond to each subset U_i and denote the text $\sigma_0(x)\sigma_1(x)\ldots\sigma_n(x)\ldots$ defined by the relations

$$\sigma_n(x) = a_{i_n} \quad \text{if } f_{n\tau}(x) \in U_{i_n}$$

by $\mathbf{T}(x, \tau)$.

Note, that if $M = 1$ then introduced texts coincide with the sturmian beams generated by the mapping (9). The principal result to be proved in the paper indicates that a direct analog of Theorem 3 for multidimensional tori shifts is not valid:

Theorem 4. *The text $\mathbf{T}(y, \tau)$ is not $\mathbf{T}(x, \tau)$-fragmentary for almost all $x, y \in I^M$ and $\tau \in \mathscr{T}$.*

This result will be obtained as a corollary to another stronger (but also more cumbersome) result. We shall need some additional definitions in order to formulate this stronger result.

Given $x, \tau \in [0,1)$, let $\mathscr{D}_m$ denote the set of all words $\mathbf{T}_n(x, \tau)$, $n \geqslant m$. How well can the text of some point $y \in [0,1)$ be "coded" by words from $\mathscr{D}_m$? To solve this problem consider the text

$$\mathbf{T}(y,\tau) = \sigma_0(y)\sigma_1(y)\ldots\sigma_i(y)\ldots$$

and denote by $\mathscr{C}_m(y)$ the set of those indices i for which there are integers k_i, n_i with $0 \leqslant k_i \leqslant i \leqslant n_i$, such that the word $w_i = \sigma_{k_i}(y)\ldots\sigma_{n_i}(y)$ belongs to $\mathscr{D}_m$. Set

$$\Delta_{n,m}(y) = \frac{1}{n}\,\#\{\mathscr{C}_m(y) \cap [0, n-m)\},$$

where $\#(X)$ is the number of elements of the set X. Then, clearly,

$$(k+n)\Delta_{k+n,m}(y) \geqslant k\Delta_{k,m}(y) + n\Delta_{n,m}(y)$$

and hence that

$$(k+n)(1 - \Delta_{k+n,m}(y)) \leqslant k(1 - \Delta_{k,m}(y)) + n(1 - \Delta_{n,m}(y)).$$

From the latter inequality the existence of $\lim_{n\to\infty}(1 - \Delta_{n,m}(y))$ follows. Then the limit $\Delta_m(y) = \lim_{n\to\infty}\Delta_{n,m}(y)$ also exists.

Theorem 5. *If $M \geqslant 3$, then $\lim_{m\to\infty}\Delta_m(y) = 0$ for almost all $x, y \in I^M$, and $\tau \in [0,1)$.*

Theorem 4 follows immediately from Theorem 5.

We remark that in view of Theorem 3 $\Delta_m(y) = 1$ for any m in one-dimensional case. In fact, the statement of Theorem 3 is even stronger than this equality.

2.3. Remark

We suspect that a similar result will also hold for the case $M = 2$. If the below proof is any guide, its proof will, however, be complicated by the problem of small denominators.

3. Proof of Theorem 5

3.1. Auxiliary results

To prove Theorem 5 we shall need some auxiliary results. For $i = 1, 2, \ldots, M$ denote by L_i the hyperplanes

$$L_1 = \{x \mid x_1 = \tau_1\},\ L_2 = \{x \mid x_2 = \tau_2\},\ \ldots,\ L_M = \{x \mid x_M = \tau_M\}.$$

Let $x \in I^M$ and let Ω be some region in I^M containing the point x and belonging to a particular subset $U_i \in \mathscr{U}$. Denote $\Omega_0 = \Omega$ and define recursively

$$\Omega_n(x) = f_\tau(\Omega_{n-1}) \cap U_{i_n},$$

where U_{i_n} is that set in $\mathscr{U}$ which contains the point $f_\tau^n(x)$ (see Fig. 2).

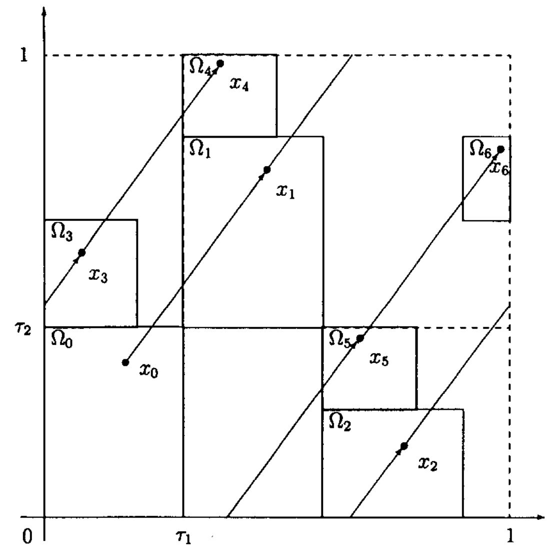

Fig. 2. Sets $\{\Omega_i\}$ for multi-dimensional shift mapping.

Since $x \in \Omega$, the set $\Omega_n(x)$ is nonempty and belongs to a single set from $\mathscr{U}$ for any n. Writing

$$\Theta_n(x) = f_\tau^{-n}(\Omega_n(x)),$$

it is clear that

$$\Theta_k(x) \subseteq \Theta_l(x) \quad \text{for } k \geqslant l \tag{12}$$

and that

$$f_\tau^k(\Theta_n(x)) \subseteq \Omega_k(x). \tag{13}$$

Hence the interior of each set $f_\tau^k(\Theta_n(x))$, $k = 0, 1, \ldots, n$, will not intersect with any of the hyperplanes L_i, $i = 1, 2, \ldots, M$.

Lemma 4. *For* $z \in I^M$

$$\sigma_0(z)\sigma_1(z)\ldots\sigma_k(z) \in \mathscr{D}_m(x) \tag{14}$$

if and only if $k \geqslant m$ *and*

$$z \in \Theta_k(x). \tag{15}$$

Proof. Suppose that (15) holds. Since $f_\tau^i(z) \in f_\tau^i(\Theta_k(x)) \subseteq \Omega_i(x)$ (see (13)) and $f_\tau^i(x) \in \Omega_i(x)$, then

$$\sigma_i(z) = \sigma_i(x), \quad i = 0, 1, \ldots, k,$$

and inclusion (14) follows.

Now, suppose that inclusion (14) is valid. Then, by definition of sets $\{\Omega_i(x)\}$, the inclusion $f_\tau^k(z) \in \Omega_k(x)$ holds. Hence $z \in f_\tau^{-k}(\Omega_k(x)) = \Theta_k(x)$, which is inclusion (15). □

Lemma 5. *If* $j \in \mathscr{C}_m(y)$ *then*

$$f_\tau^k(y) \in \left\{ \bigcup_{i=0}^{m-1} f_\tau^i(\Theta_m(x)) \right\} \cup \left\{ \bigcup_{i=m}^{\infty} \Omega_i(x) \right\}. \tag{16}$$

Proof. If $j \in \mathscr{C}_m(y)$ then by definition of the set $\mathscr{C}_m$ there exist integers k and n with $k \leqslant j \leqslant n$ and $n \geqslant k + m$, such that

$$\sigma_k(y) \ldots \sigma_j(y) \ldots \sigma_n(y) \in \mathscr{D}_m.$$

In addition for $z = f_\tau^k(y)$ the equalities

$$\sigma_{k+i}(y) = \sigma_i(z), \quad i = 0, 1, \ldots, n-k,$$

are valid. Then, by virtue of Lemma 4, $z \in \Theta_{n-k}(x)$. Since $n - k \geqslant m$, from this inclusion and (12) follows the inclusion $z \in \Theta_{n-k}(x)$. Therefore

$$f_\tau^j(y) = f_\tau^{j-k}(f_\tau^k(y)) = f_\tau^{j-k}(z) \in f_\tau^{j-k}(\Theta_m(x)).$$

If $0 \leqslant j - k \leqslant m$, then

$$f_\tau^j(y) \in \bigcup_{i=0}^{m} f_\tau(\Theta_m(x)) \tag{17}$$

and if $j - k > m$, then $f_\tau^{j-k}(\Theta_m(x)) \subseteq \Omega_{j-k}(x)$ in view of (13) and hence

$$f_\tau^j(y) \in \bigcup_{j=m}^{\infty} \Omega_j(x) \tag{18}$$

(16) then follows from (17) and (18). □

Let us now make a crucial observation. As is well known [2] the mapping $f_\tau(\cdot)$ is ergodic for any $\tau = \{\tau_1, \tau_2, \ldots, \tau_M\}$ with irrational $\tau_1, \tau_2, \ldots, \tau_M$. Hence for almost any $y \in I^M$, the value $\delta_m(y)$, which by Lemma 5, is the mean absorption time of iterations $f_\tau^i(y)$, $i = 0, 1, 2, \ldots$, into the set

$$\left\{ \bigcup_{i=0}^{m-1} f_\tau^i(\Theta_m(x)) \right\} \cup \left\{ \bigcup_{i=m}^{\infty} \Omega_i(x) \right\},$$

coincides with the Lebesgue measure of this set, that is,

$$\Delta_m(y) = \sum_{i=0}^{m-1} \operatorname{mes} \Theta_m(x) + \sum_{i=m}^{\infty} \operatorname{mes} \Omega_i(x).$$

Now the mapping f_τ is measure preserving, so

$$\sum_{i=0}^{m-1} \operatorname{mes} \Theta_m(x) = m \operatorname{mes} \Theta_m(x) = m \operatorname{mes} \Omega_m(x)$$

and hence

$$\Delta_m(y) = m \operatorname{mes} \Omega_m(x) + \sum_{i=m}^{\infty} \operatorname{mes} \Omega_i(x). \tag{19}$$

Now we are able to pose the main problem in the proof of Theorem 5:
Show that if $M \geqslant 3$ then

$$\Delta_m(y) = m \operatorname{mes} \Omega_m(x) + \sum_{i=m}^{\infty} \operatorname{mes} \Omega_i(x) \to 0 \quad \text{as } m \to \infty. \tag{20}$$

3.2. The one-dimensional case revisted

To solve the main problem stated above we shall consider only the case where $0 < x_i < \tau_i$, $i = 1, 2, \ldots, M$, for which we take

$$\Omega = [0, \tau_1) \times [0, \tau_2) \times \cdots \times [0, \tau_M).$$

This set Ω is the maximal set containing x and contained in a single subset from $\mathscr{U}$. It is obvious that for any i the set $\Omega_i(x)$ is parallelepiped, i.e.

$$\Omega_i(x) = [a_{i1}, b_{i1}) \times [a_{i2}, b_{i2}) \times \cdots \times [a_{iM}, b_{iM}).$$

Let us determine upper bounds for the lengths of sides of parallelepiped $\Omega_i(x)$. Clearly, it suffices to do this just for the first side $\omega_{i1} = [a_{i1}, b_{i1})$.

Consider one-dimensional shift mapping $f_\tau(x)$, let $\omega_0 = \omega = [0, \tau)$, and define

$$\omega_n = \begin{cases} f_\tau(\omega_{n-1}) \cap [0, \tau) & \text{if } f_{n\tau}(x) \in [0, \tau), \\ f_\tau(\omega_{n-1}) \cap [\tau, 1) & \text{if } f_{n\tau}(x) \in [\tau, 1). \end{cases}$$

Then for each n the set ω_n is an interval. If we write $\theta_n = f_\tau^{-n}(\omega_n)$, then $\omega_n = f_\tau^n(\theta_n)$. Let $n_0 = 0$ and successively choose the integer n_i as the smallest integer $n > n_{i-1}$ satisfying the condition $\theta_n \neq \theta_{n_i - 1}$. Then

$$\theta_0 \supset \theta_1 \supset \cdots \supset \theta_i \supset \theta_{i+1} \cdots.$$

Lemma 6. *For any $n_i \leqslant n < n_{i+1}$ the equalities $\theta_n = \theta_{n_i}$ are valid, one of the endpoints of the interval ω_{n_i} is either 0 or τ and neither of these points belongs to the interior of the intervals $S_\tau^n(\theta_i)$ for $n = 0, 1, \ldots, n_{i+1} - 1$.*

Let $\{p_n/q_n\}$ denotes the convergent sequence of the simple continued fraction (see, e.g., [6]) of the number τ defined by the condition $p_0 = 0$, $q_0 = 1$.

Lemma 7. *For almost all τ and for any $\varepsilon > 0$ there is an integer $K = K(\tau, \varepsilon)$ such that*

$$q_{n+1} < q_n^{1+\varepsilon} \quad \text{for } n > K. \tag{21}$$

Proof. According to Theorem 4 on p. 164 of [2], for almost all τ there exists $c = c(\tau) > 1$ such that

$$q_n^{1/n} \to c \quad \text{for } n \to \infty.$$

Hence

$$\frac{q_{n+1}^{1/(n+1)}}{(q_n^{1+\varepsilon})^{1/n}} \to c^{-\varepsilon} \quad \text{for } n \to \infty,$$

so

$$\frac{q_{n+1}}{q_n^{1+\varepsilon}(q_n^{1+\varepsilon})^{1/n}} - c^{-\varepsilon(n+1)} \to 0 \quad \text{for } n \to \infty$$

and

$$\frac{q_{n+1}}{q_n^{1+\varepsilon}} \to 0 \quad \text{for } n \to \infty$$

hold. The required inequality (21) is thus valid for all sufficiently large values of n. $\square$

Lemma 8. *Let $\xi = [z, z + \eta) \subseteq [0, 1)$ and let N be such that $f_\tau^n(\xi) \cap \{0\} \neq \emptyset$ for $0 \leqslant n < N$. Then*

$$\eta < \frac{2}{N^{1/(2+\varepsilon)}} \tag{22}$$

for almost all τ and for any $\varepsilon > 0$.

Proof. Define $\xi_0 = \xi$ and $\xi_i = f_\tau^i(\xi)$ for $i = 1, 2, \ldots$ There is an alternative: either all of intervals ξ_i are pairwise nonintersecting or there is a such minimal k for which $\xi_k \cap \xi_0 \neq \emptyset$.

In the first case the total length of the intervals ξ_i, $i = 0, 1, \ldots, N - 1$ does not exceed 1. Since the shift mapping f_τ is measure-preserving, the lengths of all intervals ξ_i are then identical and equal to η. Therefore

$$\eta \leqslant \frac{1}{N}$$

and the required estimate (22) holds for any $\varepsilon > 0$.

In the second case a more detailed analysis is required. Introduce the intervals $\zeta_i = \xi_i - \{z\}$, $i = 1, 2, \ldots$ From the identity

$$f_\tau(x + z) \equiv f_\tau(x) + z \pmod 1, \tag{23}$$

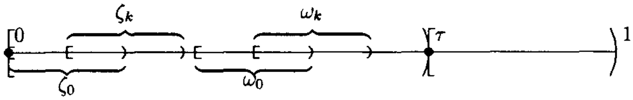

Fig. 3. Relation between sets ω_i and ζ_i.

it then follows that

$$\zeta_i = f_\tau^i(\zeta_0), \quad i = 1, 2, \ldots,$$

with

$$\zeta_k \cap \zeta_0 \neq \emptyset, \qquad \zeta_i \cap \zeta_0 = \emptyset \quad \text{for } i = 1, 2, \ldots, k-1. \tag{24}$$

In view of (24)

$$|f_\tau^k(0)| < \eta \text{ or } |f_\tau^k(0) - 1| < \eta. \tag{25}$$

(Fig. 3 corresponds to the first case). According to property of the best approximation for convergent sequence of continued fractions (see, e.g. [6]) the integer k coincides with one of numbers $\{q_n\}$, say $k = q_m$. In both cases (25) $|\tau q_m - p_m| < \eta$ and hence

$$\left| \tau - \frac{p_m}{q_m} \right| < \frac{\eta}{q_m}. \tag{26}$$

At the same time (see, e.g. [6])

$$\frac{1}{2q_m q_{m+1}} < \left| \tau - \frac{p_m}{q_m} \right| . \tag{27}$$

On the other hand for $k < q_m$ there are no points of the form $f_\tau^k(0)$ in the intervals $[0, \eta)$ and $[1 - \eta, 1)$. Since $f_\tau^{q_{m-1}}(0) = \tau q_{m-1} + p_{m-1} \pmod 1$, then $|\tau q_{m-1} + p_{m-1}| > \eta$ and therefore

$$\frac{\eta}{q_{m-1}} < \left| \tau - \frac{p_{m-1}}{q_{m-1}} \right| < \frac{1}{q_{m-1} q_m}. \tag{28}$$

Combining (26)–(28) we obtain

$$\frac{1}{2q_{m+1}} < \eta < \frac{1}{q_m}, \qquad k = q_m. \tag{29}$$

Now from the definition of the intervals $\{\zeta_n\}$ and from identity (23) it follows that lower endpoints of the intervals ω_0 and ω_k differ by $|\tau q_m - p_m| > 1/2q_m q_{m+1}$. Hence, applying the mapping $f_\tau 2q_m q_{m+1}$ times to the interval ω_0, we can cover the whole interval $[0, 1)$ and in particular the point 0. Therefore

$$N \leqslant 2q_m q_{m+1} < 2q_{m+1}^2.$$

Now from Lemma 7 it follows $q_{m+1} < q_m^{1+\varepsilon}$ for m sufficiently large, so

$$N < 2q_m^{2+\varepsilon}.$$

Applying the right inequality (29) we obtain

$$N < \frac{2}{\eta^{2+\varepsilon}}$$

and hence

$$\eta < \frac{2^{1/(2+\varepsilon)}}{N^{1/(2+\varepsilon)}} < \frac{2}{N^{1/(2+\varepsilon)}},$$

which completes the proof of Lemma 8. □

3.3. *Proof of Theorem 5*

As was shown in Section 3.1, from Lemma 5 it follows that in order to prove Theorem 5 we need only establish relation (20). But from Lemma 8 and the definition of sets $\Omega_i(y)$ for almost all τ the following estimate is valid:

$$\operatorname{mes} \Omega_i(y) \leqslant \frac{2^M}{i^{M/(2+\varepsilon)}}.$$

Hence from (19)

$$\Delta_m(y) \leqslant \frac{m2^M}{m^{M/(2+\varepsilon)}} + \sum_{i=m}^{\infty} \frac{2^M}{i^{M/(2+\varepsilon)}}$$

or, what is the same,

$$\Delta_m(y) \leqslant 2^M m^{1-M/(2+\varepsilon)} + 2^M \sum_{i=m}^{\infty} i^{-M/(2+\varepsilon)}.$$

Note, that the value of ε can be chosen arbitrarily small. Hence, the right-hand part of the latter inequality clearly tends to 0 as $m \to \infty$ when $M \geqslant 3$. This completes the proof of Theorem 5.

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